Int. J. Heat Mass Transfer. Vol. 22, pp. 771–774
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0017-9310/79/0501-0771 $02.00/0

SHORTER COMMUNICATIONS

EXTENSION OF A MODIFIED INTEGRAL METHOD TO BOUNDARY CONDITIONS OF PRESCRIBED SURFACE HEAT FLUX

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NOMENCLATURE

C_p, constant pressure specific heat;
DIM, SIM, double and single integral methods, respectively;
k, thermal conductivity;
n, wedge flow parameter;
Nu, Pr, Re, Nusselt, Prandtl and Reynolds numbers, respectively;
q_w, surface heat flux;
r, aerodynamic recovery factor;
t, time;
u, u_s, v, local velocity in x direction, free stream and y direction, respectively;
x, y, space coordinates along and perpendicular to surface, respectively.

Greek symbols

α, thermal diffusivity;
ε, error vector;
θ, temperature excess above free stream;
ξ, ω, dummy variables;
δ, δ_t, velocity and thermal boundary-layer thicknesses, respectively;
ν, kinematic viscosity.

Subscripts

s, ss, free stream and eventual steady state, respectively;
w, wall value;
x, y, evaluated at positions x and y, respectively.

INTRODUCTION

VOLKOV [1] reasoned that the greatest portion of the error incurred in the use of low order integral methods, such as the Karman–Pohlhausen type which will be referred to as the single integral method (SIM), is caused by the need to directly differentiate the approximating sequence at a domain boundary. To circumvent this difficulty, he suggested viewing the usual SIM equation as a relation giving the required derivative at the boundary without explicitly differentiating the sequence. The parameter function remaining in the approximating sequence is then found by a second integration of a general SIM equation. For brevity, it is proposed to refer to Volkov's overall procedure as the double integral method (DIM). In [1] where the DIM is applied to some simple boundary-layer flow problems, and in [2], where Volkov and Li-Orlov apply it to a transient heat conduction problem, it is found that the DIM yields very good results, particularly for the derivatives of the approximating sequence at the boundary which are needed to predict the wall shear stress and the heat flux. Parenthetically, it may be mentioned that Bromley [3] had used the DIM much earlier (1952) in the solution of the problem of laminar film condensation on a horizontal tube. Bromley not only used the DIM, though without making any comments about the appealing and important features of his technique or the possibility of generalizing it to other problems, but also showed how one could, easily and rationally, generate improvements of the original approximating sequence from a general SIM equation, a step also suggested by Volkov and Li-Orlov in [2]. More recently, Zien, in a series of works [4–6] extols the merits of the DIM in boundary-layer flow and heat-transfer problems involving suction or blowing. Zien compares his DIM results with SIM results and with exact analytical solutions, when available, and finds, generally, that the DIM, in its prediction of skin friction coefficients and local Nusselt numbers, exhibits good agreement with the exact solutions and outperforms the SIM for the same approximating sequences. Furthermore, Zien finds that relative simplicity of application of the DIM accompanies the rather high accuracy achieved because very elementary approximating sequences were used. He finds that linear velocity profiles in [4] and [5], and a linear temperature profile in [6], perform about as well, in the DIM, as do quartic profiles of velocity and of temperature. In [7], Zien makes use of one aspect of the DIM in the solution to transient heat conduction problems.

The previous works, in line with Volkov's original idea of avoiding direct differentiation of the approximating sequence at the domain boundary, treat only cases where the dependent variables are specified along the boundary with their derivatives constituting unknowns of interest and importance. The only exception to this is in Zien [7] where the boundary heat flux is specified in a conduction problem, but the solution procedure is not the DIM, but is more properly classified as a two parameter method of moments. It is the intent of the present work to give a rationale for expecting the DIM to be equally applicable to problems where the boundary derivative is a known, given quantity, and to show the application of the DIM to some representative problems of this type, namely, a simple steady state forced convection problem, an aerodynamic heating problem, and a transient forced convection problem. Also, one of the basic concepts of the DIM is extended to evaluation of derivatives within the domain itself when they occur in the governing DIM equation.

ANALYSIS

Consider steady, laminar, constant property, low speed, two dimensional planar, boundary-layer type flow for which the thermal energy equation has the following form:

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} - \alpha\frac{\partial^2 \theta}{\partial y^2} = 0. \quad (1)$$

When it is desired to predict the surface heat flux, the local Nusselt number is often formed from the solution to (1).

$$Nu_x = -\frac{x}{\theta_w}\left(\frac{\partial \theta}{\partial y}\right)_w. \quad (2)$$

The usual SIM equation is found either by an integration of (1) over the thermal boundary layer or by making an energy balance on the thermal boundary layer yielding,

$$\frac{d}{dx}\int_0^{\delta_t} u\theta\,dy + \alpha\left(\frac{\partial \theta}{\partial y}\right)_w = 0. \quad (3)$$

If one integrates equation (1) from zero to y and then integrates the resulting equation from zero to δ_t, there results

$$\int_0^{\delta_t}\left[\frac{\partial}{\partial x}\int_0^y u\theta\,d\xi\right]dy-\int_0^{\delta_t}\left[\theta_y\frac{\partial}{\partial x}\int_0^y u\,d\xi\right]dy+\alpha\theta_w+\alpha\left(\frac{\partial\theta}{\partial y}\right)_w\delta_t=0. \quad (4)$$

Next one avoids direct differentiation of the approximating sequence at the boundary by solving for $(\partial\theta/\partial y)_w$ from (3) and inserting it into (4) to give the DIM equation. Now, in the Karman–Pohlhausen SIM, equation (3) represents an attempt to make the error or residual vector [the LHS of (1) with the approximating sequences inserted] in function space small by requiring that its component in the "1" abstract direction be zero by virtue of the inner product (3) being set equal to zero. Or, physically, since the error vector in the SIM is the energy imbalance per unit volume, (3) represents an attempt to minimize this imbalance. The DIM equation (4) can be interpreted in much the same way except the error vector in function space is now given by

$$\varepsilon=\frac{\partial}{\partial x}\int_0^y u\theta\,d\xi-\theta_y\frac{\partial}{\partial x}\int_0^y u\,d\xi-\alpha\frac{\partial\theta}{\partial y}+\alpha\left(\frac{\partial\theta}{\partial y}\right)_w. \quad (5)$$

once the approximating sequences are inserted on the right of (5) with the last term replaced by its equivalent from equation (3). Then δ_t is determined by requiring that the inner product, $\varepsilon\cdot 1$, be zero which is, of course, equation (4). Physically, equation (5) represents the energy imbalance of a general control volume within the thermal boundary layer and, hence, equation (4) is an attempt to minimize the energy imbalance of this general, finite in y, control volume. Regardless of whether one views equation (4) from the standpoint of requiring a particular component of the error vector in function space to be zero or from the physical standpoint, it seems reasonable to expect it, and the DIM, to work even when no unknown derivative appears on the boundary, that is, even when $(\partial\theta/\partial y)_w$ is a given quantity. If this is the case, then one does not use equation (3) to solve for $(\partial\theta/\partial y)_w$, but, rather, just inserts the known value of the boundary derivative into equation (4) and solves.

To investigate the accuracy of the DIM in problems where the boundary derivative is specified, it was decided to apply it to some representative forced convection problems with specified surface heat flux.

1. *Constant flux flat plate*

Considered first was the simple case of a flat plate with constant surface heat flux. The following Kantorovich approximating sequences were used:

$$\frac{u}{u_s}=\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\frac{y^3}{\delta^3} \quad (6)$$

$$\frac{\theta}{q_w\delta_t/k}=\frac{2}{3}-\frac{y}{\delta_t}+\frac{y^3}{3\delta_t^3}. \quad (7)$$

Inserting (6) and (7) into (3) and into (4) yields the SIM and DIM solutions, respectively. The surface temperature from these solutions, as well as the exact value from Kays [8], is given as follows with per cent error shown in the parentheses.

Table 1. Comparison between SIM, DIM, and exact result for flat plate with constant surface heat flux

$\frac{\theta_w Re_x^{1/2}Pr^{1/3}}{q_w x/k}$		
Exact [8]	SIM	DIM
2.2075	2.432(10.2%)	2.167(1.82%)

As can be seen from Table 1, the DIM gives a dramatic improvement of results over the SIM even though direct differentiation of the approximating sequence is not needed in either method for the specified flux boundary condition.

2. *Aerodynamic heating of a flat plate*

Dealt with next is high speed flow over an insulated flat plate and it is desired to predict the aerodynamic recovery factor r by both SIM and DIM. Mathematically, the problem description becomes,

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}-\alpha\frac{\partial^2\theta}{\partial y^2}-\frac{\nu}{C_P}\left(\frac{\partial u}{\partial y}\right)^2=0 \quad (8)$$

with $\theta\to 0$ as $y\to\infty$ and $(\partial\theta/\partial y)_w=0$.

The SIM and DIM equations (3) and (4) are modified now by the inclusion of the appropriate integrations of the last term displayed in (8) and by $(\partial\theta/\partial y)_w$ being zero. Equation (6) was used as the approximate profile for velocity while that for the temperature is as follows:

$$\frac{\theta}{\frac{\delta_t^2 Pr}{2C_P}\left(\frac{\partial u}{\partial y}\right)_w^2}=\frac{1}{3}-\frac{y^2}{\delta_t^2}+\frac{2}{3}\frac{y^3}{\delta_t^3}. \quad (9)$$

It is noticed that the velocity derivative at the wall is needed in (9) while the velocity derivative within the domain is needed in the DIM equation because of the last term in (8). The DIM avoids direct differentiation of the profile at the boundary by using, for the flat plate,

$$\left(\frac{\partial u}{\partial y}\right)_w=\frac{1}{\nu}\frac{d}{dx}\int_0^\delta u(u_s-u)\,dy. \quad (10)$$

Next, it is proposed to avoid direct differentiation within the domain by utilizing (10) in conjunction with a general SIM equation (integrated to y instead of δ), namely,

$$\frac{\partial u}{\partial y}=\left(\frac{\partial u}{\partial y}\right)_w-\frac{u}{\nu}\frac{\partial}{\partial x}\int_0^y u\,d\xi+\frac{1}{\nu}\frac{\partial}{\partial x}\int_0^y u^2\,d\xi. \quad (11)$$

For the SIM, the $\partial u/\partial y$ and $(\partial u/\partial y)_w$ were found by differentiating equation (6).

Under the condition of $\delta_t\leqslant\delta$, the recovery factor r was found by SIM and DIM and compared to Pohlhausen's results, as reproduced in Schlichting [9], in Table 2. Per cent errors are also shown.

For the Prandtl number values of 0.7 and 1.0, the condition that $\delta_t\leqslant\delta$ was violated by a slight amount. This was not considered severe enough to make significant difference in r, however, because equation (6) gave $0.942u_s$ as the velocity at the edge of the thermal boundary layer for $N_{Pr}=0.7$ and this was adjudged close enough to the correct value of u_s, especially since θ between δ and δ_t at $N_{Pr}=0.7$ is so close to zero that the vast bulk of the integral is accumulated before δ is reached. In addition, Illingworth [10], using the same profiles for u and θ, but considering that $\delta_t\geqslant\delta$, arrives at $r=0.907$ by the SIM rather the value of $r=0.9082$ calculated here. The difference is too slight to warrant recalculation for these two Prandtl numbers.

Table 2 indicates the decided superiority of the DIM over the SIM in this zero surface heat flux problem.

Table 2. Comparison between SIM, DIM, and exact result for aerodynamic heating of flat plate

		$r=\frac{\theta_w}{u_s^2/2C_P}$	
Pr	Exact (from [9])	SIM	DIM
0.7	0.835	0.7795 (6.7%)	0.8686 (4.0%)
1.0	1.0	0.9082 (9.2%)	1.0458 (4.6%)
7.0	2.515	2.225 (11.5%)	2.610 (3.8%)
10.0	2.965	2.584 (14.7%)	3.021 (1.9%)
15.0	3.535	3.045 (13.9%)	3.544 (0.2%)

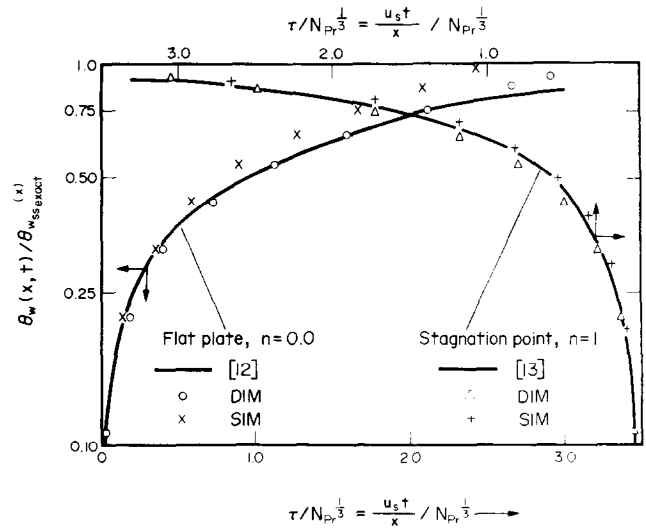

FIG. 1. Comparison of transient wall temperature excess ratios predicted by SIM, DIM, and [12] and [13].

Transient forced convection for flat plate and stagnation point

Considered next was a forced convection problem, with steady velocity field and wedge flow free stream velocity, in which a fluid is flowing over a body and the initial temperature excess is zero throughout when, suddenly, the body experiences a step change in surface heat flux from zero to q_w. This initiates a transient temperature profile within the flowing fluid and it is desired to predict the time varying body surface temperature. The governing DIM equation can be shown to be,

$$\int_0^{\delta_t}\left[\frac{\partial}{\partial x}\int_0^y u\theta\,d\xi\right]dy-\int_0^{\delta_t}\theta_y\left[\frac{\partial}{\partial x}\int_0^y u\,d\xi\right]dy+\int_0^{\delta_t}\left[\int_0^y\frac{\partial\theta}{\partial t}d\xi\right]dy+\alpha\theta_w-\frac{\alpha q_w\delta_t}{k}=0. \quad (12)$$

The approximating sequences used were as follows:

$$\frac{u}{u_s(x)}=\frac{4}{3}\frac{y}{\delta}-\frac{1}{3}\frac{y^4}{\delta^4} \quad (13)$$

$$\frac{\theta}{q_w\delta_t/k}=\frac{1}{2}-\frac{y}{\delta_t}+\frac{y^2}{2\delta_t^2} \quad (14)$$

δ_t is now a function of both x and time t. The profiles of u and θ were dictated by the choice made by Sucec [11] for easy comparison. Details of the SIM solution to this problem are given in [11]. Considering wedge flows, $u_s = Ax^n$, requiring that $\delta_t \leqslant \delta$, and defining new variables for convenience, namely, $K = 90\alpha/(5-n)$, $R_f = [\delta_t/\delta_{tss}]^3$, and $\tau = u_s t/x$, insertion of equations (13) and (14) into (12) gives

$$\frac{\partial R_f}{\partial x}+\left[\frac{\left(\frac{75}{28}\right)\left(\frac{34\nu}{5n+1}\right)^{1/3}}{K^{1/3}R_f^{1/3}x}+\frac{(n-1)\tau}{x}\right]\frac{\partial R_f}{\partial\tau}=\frac{3(5-n)[1-R_f]}{14x}. \quad (15)$$

Equation (15) is now solved by the method of characteristics, as outlined for a similar equation in [11], to yield, as the DIM solution,

$$\frac{\tau}{Pr^{1/3}}=\left(\frac{75}{6}\right)\left(\frac{34}{90}\right)^{1/3}\frac{[1-R_f]^{14(1-n)/3(5-n)}}{(5n+1)^{1/3}(5-n)^{2/3}}\times\int_1^{1/(1-R_f)}\frac{\omega 4(1-3n)/3(5-n)\,d\omega}{(\omega-1)^{1/3}}. \quad (16)$$

After performing the numerical integration in (16) and normalizing the surface temperature by the steady state values contained in Chao and Cheema [12] for $n = 0$, the flat plate, and in Chao and Jeng [13] for $n = 1$, the stagnation point, the ratio $\theta_w(x,t)/\theta_{w_{ss\,exact}}$ is plotted in Fig. 1 for the DIM and SIM and also for the more nearly exact results of [12] and [13]. For the flat plate, the DIM yields significant improvement over the SIM, while for the stagnation point the conclusion is not so clearcut since both methods give comparable accuracy.

CONCLUDING REMARKS

The basic aims of this work are to show why one would reasonably expect the DIM to be applicable even to problems in which there are no unknown boundary first derivatives, to demonstrate the use of the DIM in a problem where a derivative within the solution domain appears in the DIM equation itself, and to display the advantage, in increased accuracy, enjoyed by the DIM over the SIM in this type of problem.

The results of the three calculations performed using the DIM in these specified surface flux problems are extremely encouraging. It appears as if the accuracy and simplicity of the DIM as noted by Zien [6], in connection with cases of unknown first derivatives at domain boundaries, carry over, to a smaller degree, to the class of problems where these derivatives are known in advance.

Although not reported herein, the author has also applied the DIM to the other type of transient convection problem reported in [11], namely, the step change in surface temperature. Increased accuracy of the DIM over the SIM and relative insensitivity of DIM results to profile shape (even linear ones were used for u and θ) were found for these transient problems that parallel the observations of Zien [6] for steady state convection.

It may also be of interest to know that the DIM applied to a natural convection heat-transfer problem leads to a very weak solution, namely, one for a fluid whose shear stress is zero within the domain. This behavior is the manifestation of a term, in the DIM error vector ε, being automatically orthogonal to the "1" vector. The term referred to, namely $\nu(\partial u/\partial y)$, appears in the momentum equation and is readily seen to yield zero when integrated between $y = 0$ and $y = \delta$. This occurrence is not a fault of the method since this can also be observed for the SIM on other problems. The remedy, in both methods, is to increase the number of unknown parameter functions from one to two or more and appropriately employ additional weighting functions.

Acknowledgement—A portion of this material is based upon work supported by the National Science Foundation under Grant No. ENG-77-18268 and the author thanks them for their support.

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Int. J. Heat Mass Transfer. Vol. 22, pp. 774–776
0017 9310/79/0501-0774 $02.00/0

ON PREDICTING BOILING BURNOUT FOR HEATERS COOLED BY LIQUID JETS*

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(*Received* 12 *June* 1978 *and in revised form* 10 *October* 1978)

NOMENCLATURE

- A, area;
- a, exponent expressing We_f dependence of δ;
- D, disc diameter;
- d, jet diameter;
- $\dot{E}$, $\dot{K.E.}$, rate of energy; rate of kinetic energy;
- h_{fg}, latent heat of vaporization;
- q_{max}, peak (or "burnout") boiling heat flux;
- r, ρ_f/ρ_g;
- u_f, liquid velocity;
- $v_g, v_{g_{max}}$, vapor velocity; maximum vapor velocity; $q_{max}/\rho_g h_{fg}$;
- We_f, "liquid" Weber number, $\rho_f u_f^2 D/\sigma$.

Greek symbols

- α, fraction of liquid flow directed into spray;
- β, D/δ;
- δ, surface area average droplet diameter;
- ρ_f, ρ_g, saturated liquid and vapor densities;
- σ, surface tension;
- ϕ, $(v_{g_{max}}/u_f) = q_{max}/\rho_g h_{fg} u_f$.

INTRODUCTION

MONDE and Katto [1] recently provided an extremely successful correlation of peak heat flux (q_{max}) data for saturated liquid jets impinging on discs. Their configuration is shown in Fig. 1. They correlated 93% of 150 original and previous [2] q_{max} data for water and Freon 113, within ±25% of the best line through them. The correlation can be written as:

$$\phi = 0.0745 r^{0.725}/We_f^{1/3} \tag{1}$$

*We are pleased to acknowledge support for this work under NSF Grant ENG77-25029.

where ϕ is the vapor escape velocity at the peak heat flux, $v_{g_{max}}$, divided by the jet velocity, u_f. Thus $\phi \equiv q_{max}/\rho_g h_{fg} u_f$. The ratio of the saturated liquid and vapor densities is designated as r and $We_f \equiv \rho_f u_f^2 D/\sigma$, where D is the diameter of the heater.

More recently Katto and Ishii [3] achieved comparable success in correlating data for plane jets impinging at a 15° angle to a square heater. This time they based We_f on the length of the plate and found:

$$\phi = 0.0164 r^{0.867}/We_f^{1/3}. \tag{2}$$

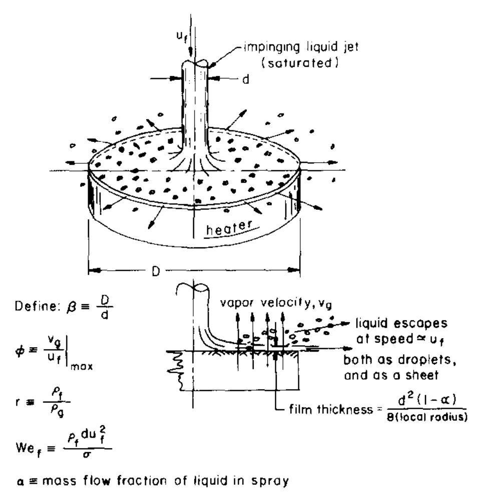

FIG. 1. Jet and heater configuration with nomenclature.